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Is There a Jordan Geometry Underlying Quantum Physics?

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Abstract There have been several propositions for a geometric and essentially non-linear formulation of quantum mechanics, see, e.g., (Ashtekar and Schilling, in On Einstein's Path, Springer, Berlin, 1998; Brody and Hughston, J. Geom. Phys. 38:19–53, 2001; Cirelli et al. J. Geom. Phys. 45:267–284, 2003; Kibble, Commun. Math. Phys. 65:189–201, 1979). From a purely mathematical side, the point of view of *Jordan algebra theory* might give new strength to such approaches: there is a "Jordan geometry" belonging to the Jordan part of the algebra of observables, in the same way as Lie groups belong to the Lie part. Both the Lie geometry and the Jordan geometry are well-adapted to describe certain features of quantum theory. We concentrate here on the mathematical description of the Jordan geometry and raise some questions concerning possible relations with foundational issues of quantum theory.

Keywords Generalized projective geometries · Jordan algebras (-triple systems, -pairs) · Quantum theory · Twistor theory

1 Introduction

Can quantum theory be based on the commutative and non-associative "Jordan product"

$$X \bullet Y = \frac{1}{2}(XY + YX)$$

alone, or do we need the associative product XY somewhere in the background? In his foundational work [25, 26], Pascual Jordan gives an affirmative answer to the first question. From a more contemporary perspective, E. Alfsen and F.E. Schultz write ([1], p. vii): "…it has been proposed to model quantum mechanics on Jordan algebras rather than on associative algebras [27]. This approach is corroborated by the fact that many physically relevant

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properties of observables are adequately described by Jordan constructs. However, it is an important feature of quantum mechanics that the physical variables play a dual role, as observables *and* as generators of transformation groups. ... Therefore both the Jordan product and the Lie product of a C^*-algebra are needed for physics, and the decomposition of the associative product into its Jordan part and its Lie part separates two aspects of a physical variable."

I think that this point of view is very interesting and deserves to be developed further. From the mathematical side, the "Lie part" has so far attracted much more attention than the "Jordan part", because it has a beautiful relation with *geometry*, namely via the *Lie functor*: to every (finite or infinite dimensional) Lie group we can assign a Lie algebra, which is a sort of infinitesimal neighborhood of the origin of the group. Is there something similar for the "Jordan part"—can we find a global and geometric structure (finite or infinite dimensional) of which the Jordan algebra somehow is an infinitesimal or tangent structure? If this is so, one should expect that this structure might play a rôle in physical theories and contribute to the understanding of "the geometry of quantum mechanics" (I use this term here to embrace very different approaches such as, e.g., [2] and [42]). In fact, this was the main motivation for the author's mathematical work on Jordan structures, leading to the result that there is indeed a corresponding geometric object, introduced in [6] and called *generalized projective geometry*. The first aim of the present work is to explain this purely mathematical theory to readers coming from physics rather than from mathematics.

The second aim is to raise the question whether there ought to be relevant consequences of such a "Jordan geometric" approach for physics. However, since the author is a mathematician and not a physicist, I will only try to motivate why I think that this question may be interesting, but not to answer it: if the algebra of observables is indeed equivalent to some geometric, "global" and non-linear object, then it is possible to translate the whole formulation from the linear level into a geometric and non-linear language. As long as one restricts oneself to a faithful translation, nothing is gained, and also nothing is lost. Now, all general arguments in favour of geometric approaches, given, e.g., in [2] and [19], remain fully valid, and as explained by these authors, the geometric formulation inevitably suggests new ideas and concepts which can no longer be considered as a faithful translation of the theory we started with. In other words, at this point speculations begin. If one believes that the present formulation of quantum theory is complete, then of course one has no reason and no need for such speculations. For the sake of clarity I should admit that this is not my conviction, and I rather adhere the point of view of several authors, explained very convincingly by R. Penrose in [38], Chaps. 29 and 30, that the present formulation is not satisfactory and that there are foundational problems which are "not just matter of philosophical interest" (loc. cit., p. 865). My speculations are, to some extent, similar to those of the authors mentioned above, but in some parts they are different and, perhaps, complementary. More specifically, in the first section of this work, I describe the general features of Jordan geometries in an informal way, using terms borrowed from the language of physics and thus suggesting a hypothetical physical interpretation. The main features are:

(1) *Duality.* The mathematically important distinction between space and dual space, which also is a fundamental feature of Jordan theory (cf. the notion of a *Jordan pair*, explained in Sect. 3), should also appear in a geometric formulation of quantum theory; one may call it a duality between "bras" and "kets", or between "observables" and "observers".
(2) *Linearity.* It has been strongly emphasized that quantum mechanics is a *linear* theory—and that, if we sacrifice linearity, this should be done in a "subtle but essential way"

([2], Introduction). This is achieved by assuming a suitable *local linear structure* of our geometries.

(3) *Laws.* The various local linear structures are related among each other via algebraic laws, involving both the given geometry and its dual geometry. A *generalized projective geometry* is a pair $(\mathcal{X}^+, \mathcal{X}^-)$ of geometries that are locally linear and obey certain fundamental laws.

(4) *Polarities and energy.* In classical geometry, *polarities* are used to identify a projective geometry with its dual projective geometry. In a likewise way, suitable identifications between "observables" ($\mathcal{X}^+$) and "observers" ($\mathcal{X}^-$) in a generalized projective geometry are called *polarities*. Physically, such a polarity seems to represent *energy* or the *Hamilton operator*. However, although this interpretation matches well with the Jordan part of the usual observable "Hamilton operator", it matches less well with its Lie part, which is related to the time evolution in quantum mechanics. In other words, from a purely mathematical context we are lead to a problem looking quite similar to the problem of the coexistence of the two quantum processes, the unitary "**U**-evolution" and the "state reduction **R**" (here I use the labels introduced by R. Penrose [37, 38]).

(5) *States and non-locality.* As in non-commutative geometry, and unlike the geometric approaches [2, 16, 19], we started with observables and not with (*pure*) *states*. Nevertheless, we can associate a *geometry of states* to our geometry of observer-observables. Here, a *state* is essentially a global object of the geometry, something like a projective line in a projective space or a light ray in compactified Minkowski space; hence the feature of non-locality is built in these concepts, and we feel that the analogy with Penrose's twistor theory (cf. [38]) is not just an accident.

(6) *Geometry of special relativity.* From the point of view of Jordan theory, the geometry of special relativity and the geometry of quantum mechanics are brothers, the only difference being in dimension—the former is associated to Minkowski space (which is nothing but the Jordan algebra of Hermitian 2×2-matrices) and the latter to the Jordan algebra of Hermitian operators in an infinite dimensional Hilbert space. Therefore everything we have said so far applies as well to the geometry of special relativity (the conformal compactification of Minkowski space). In particular, pure states in the geometry of special relativity indeed lead to Penrose's twistor space.

(7) *Hermitian symmetric spaces.* On the one hand, the importance of the "complex Hermitian" structure of quantum mechanics has been emphasized, e.g., in [19] and in [38]. On the other hand, there is a well-known relation between *Hermitian symmetric spaces of non-compact type*, also known as *bounded symmetric domains*, and certain ("positive Hermitian") Jordan structures, see [40]. Our setting generalizes this correspondence in several regards. Therefore, although at a first glance it looks rather different, it has close relations to previous work of several authors relating such structures to physics, and especially to quantum mechanics (cf. [22, 41] and the extensive bibliography given in [24]).

Throughout the text, I try to illustrate all these concepts by simple examples from linear algebra, so that the reader will more easily grasp (or be able to skip) the formal mathematical definitions which are given in Sect. 4. In Sect. 3, we give a brief introduction to basic notions of Jordan theory. The main result (equivalence of categories between Jordan theory and generalized projective, resp. polar geometries) is stated in Sect. 4.6. Finally, in Sect. 5, I come back to the issue of possible relations between physics and Jordan geometry—the least one can say is that some of its features match certain requirements on possible new approaches to the foundations of quantum physics that have been put forward. Moreover, the similarity with the geometry of special relativity (item (6) above) may suggest how to carry such ideas even further, following the ideas that have lead from special to general relativity.

2 The General Geometric Framework

2.1 Duality

Not only in mathematics, but also in physics it is useful to distinguish between a vector space V and its dual space V^*, even if finally one wishes to identify them. In Jordan theory, exactly the same phenomenon occurs: it turns out to be useful to look at so-called "Jordan pairs" (V^+, V^-) instead of a single Jordan algebra V, even if one often is interested in identifying V^+ and V^- with V as sets (see Sect. 3 for the formal definitions).

Therefore we define our geometric "universe" as a *pair geometry* $(\mathcal{X}^+, \mathcal{X}^-)$; this means just that $\mathcal{X}^+$ and $\mathcal{X}^-$ are sets, which we call the *space of observables* and the *space of observers*, respectively, such that there exists a basic *transversality relation*, denoted by $\top$: a pair $(x, \alpha) \in \mathcal{X}^+ \times \mathcal{X}^-$ is called *transversal*, and we then also say that "α can observe x", and we then write $\alpha \top x$ or $x \top \alpha$, such that

(a) every observer can observe at least one observable: for all $\alpha \in \mathcal{X}^-$, there exists $x \in \mathcal{X}^+$ with $x \top \alpha$;
(b) every observable can be observed by at least one observer: for all $x \in \mathcal{X}^+$, there exists $\alpha \in \mathcal{X}^-$ with $x \top \alpha$.

Here and in the sequel, all assumptions will be such that we can turn things over: the rôle of $\mathcal{X}^+$ and $\mathcal{X}^-$ is entirely symmetric—the pair $(\mathcal{X}^-, \mathcal{X}^+)$ is a universe with the same rights as $(\mathcal{X}^+, \mathcal{X}^-)$, called its *dual universe*. Writing

$$\alpha^\top := \{x \in \mathcal{X}^+ | \alpha \top x\}$$

for the set of observables that can be observed by an observer α (the "visible world of α"), our assumption means that $\mathcal{X}^+$ is covered by such sets, and vice versa. In contrast, the set $\mathcal{X}^+ \setminus \alpha^\top$, called the *horizon* or the *infinite set of* α, may or may not be empty.

Example A familiar example of a pair geometry is given by a projective space $\mathcal{X}^+ = \mathbb{P}(W)$ and its dual projective space of hyperplanes $\mathcal{X}^-$ (which may be identified with $\mathbb{P}(W^*)$, where W is a vector space and W^* its dual space), with $x \top \alpha$ meaning that x does *not* belong to the hyperplane α; in other words, $\alpha^\top$ is the complement of the hyperplane α, and its horizon is the usual "hyperplane at infinity". In the same spirit, one may consider the *Grassmann geometry of type E and co-type F*,

$$(\mathcal{X}^+, \mathcal{X}^-) = (\mathrm{Gras}_E^F(W), \mathrm{Gras}_F^E(W)),$$

where $W = E \oplus F$ is a fixed direct sum decomposition of the vector space W, and $\mathrm{Gras}_A^B(W)$ denotes the set of all subspaces Y in W that are isomorphic to A and such that Y has some complement that is isomorphic to B. A pair $(U, V) \in \mathcal{X}^+ \times \mathcal{X}^-$ is transversal if and only if U and V are complementary subspaces: $W = U \oplus V$.

2.2 Linearity

The next structural ingredient to be added to our universe $(\mathcal{X}^+, \mathcal{X}^-; \top)$ is the *principle of linearity*: for all observers α, the visible world $\alpha^\top$ is a linear space. More precisely, fixing an arbitrary observable $o \in \alpha^\top$ as "origin" in $\alpha^\top$, we require that a structure of a linear (i.e., vector) space with origin o be given on $\alpha^\top$. By duality, the same shall hold for $o^\top$ with origin α; if such a structure is given for all transversal pairs (o, α), we say that $(\mathcal{X}^+, \mathcal{X}^-, \top)$ is equipped with a structure of *linear pair geometry*. It may happen that the underlying affine space structure on $(\alpha^\top, o)$ does not depend on the choice of the origin o; if this is always the case, the geometry is called an *affine pair geometry*. In other words, $\alpha^\top$ then canonically carries the structure of an affine space. In any case, we think of $\mathcal{X}^+$ as "modelled on the linear space $\alpha^\top$", just as usual projective space $\mathbb{RP}^n$ is modelled on usual affine space $\mathbb{R}^n$, and it is not misleading to picture $\mathcal{X}^+$ as a, finite or infinite dimensional, smooth manifold, covered by "linear chart domains" $\alpha^\top$ which in turn are indexed by $\alpha \in \mathcal{X}^-$.

Example Consider the Grassmann-geometry $(\mathrm{Gras}_E^F(W), \mathrm{Gras}_F^E(W))$: it is a well-known exercise in linear algebra that the set of complements of a given subspace carries canonically the structure of an affine space, modelled on the linear spaces of linear operators $\mathrm{Hom}(F, E)$, respectively $\mathrm{Hom}(F, E)$. Thus $(\mathrm{Gras}_E^F(W), \mathrm{Gras}_F^E(W))$ is an affine pair geometry.

More generally, one may consider the geometry $(\mathrm{Flag}_{\mathfrak{e}}^{\mathfrak{f}}, \mathrm{Flag}_{\mathfrak{f}}^{\mathfrak{e}})$ of all flags in W of a given "type" and "cotype", equipped with a natural transversality relation; this defines a linear pair geometry which, however, is no longer affine in general (cf. [12]).

2.2.1 *Time*

When we speak about vector spaces, we must specify a base field $\mathbb{K}$. We consider $\mathbb{K}$ as "time", although by no means we want by this to fix the choice to be $\mathbb{K} = \mathbb{R}$, the real base field—some may prefer a "complex time", or a "p-adic time" or yet another model (see [38], Chap. 16 for some remarks on the "base field of physics"). Personally, I prefer models in which "infinitesimal times" exist, like the ring $\mathbb{R}[\varepsilon] = \mathbb{R} \oplus \varepsilon\mathbb{R}$ $(\varepsilon^2 = 0)$ of *dual numbers*, where ε may be related in some mysterious way to the Planck time. Thus, instead of fields, we will admit also (commutative unital) base *rings*, and the term "linear space" means just "$\mathbb{K}$-module". By a fortunate coincidence of terminology, the unit group $\mathbb{K}^\times$ of $\mathbb{K}$ can then be seen as the set of possible "units of time measurement"; the non-invertible elements of $\mathbb{K}$ may be considered as "infinitesimal times", which cannot be used as units of time measurement. Whatever the structure of time be, and in contrast to a trend set by Hilbert's "Foundations of Geometry" (cf. [17]), we accept the base ring $\mathbb{K}$ as God-given, and we do not try to "reconstruct" it from incidence structures or other data.

2.2.2 *Laws*

We assume that the universe $(\mathcal{X}^+, \mathcal{X}^-)$ is a linear or, better, affine pair geometry over $\mathbb{K}$, governed by *laws*. These laws give it a certain structure, somewhat similar to the one of a projective geometry $(\mathbb{P}\mathcal{H}, \mathbb{P}\mathcal{H}^*)$ of a Hilbert space $\mathcal{H}$, but more flexible and incorporating many other situations. In a sense, these laws describe the "basic rules of communication" between various observers α, β and their visible worlds $\alpha^\top$, $\beta^\top$ which, after all, shall be interpreted by α and β as their images of the *same* world—at least, if they live sufficiently close to each other so that the common part $\alpha^\top \cap \beta^\top$ of their visible worlds is non-empty.

The formal statement of such laws, to be given in Sect. 4, is described by *identities* for the so-called *structure maps*: if $x\top\alpha$, $y\top\alpha$, $z\top\alpha$ and $r \in \mathbb{K}$, then let $r_{x,\alpha}(y) := ry$ denote the scalar multiple $r \cdot y$, and $y +_{x,\alpha} z := y + z$ the sum of y and z in the $\mathbb{K}$-module $\alpha^\top$ with zero vector x. In other words, we define maps of three (resp. four) arguments by

$$\Pi_r := \Pi_r^+ : \quad (\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+)^\top \to \mathcal{X}^+, \quad (x, \alpha, y) \mapsto \Pi_r(x, \alpha, y) := r_{x,\alpha}(y),$$

$$\Sigma := \Sigma^+ : \quad (\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ \times \mathcal{X}^+)^\top \to \mathcal{X}^+, \quad (x, \alpha, y, z) \mapsto \Sigma(x, \alpha, y, z) := y +_{x,\alpha} z,$$

where the domain of definition of Π_r is the "space of generic triples",

$$(\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+)^\top = \{(x, \alpha, y) \in \mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ \,|\, x\top\alpha,\ y\top\alpha\},$$

and the domain of Σ is the similarly defined space of generic quadruples. By duality, Π_r^- and Σ^- are defined. The structure maps encode all the information of a linear pair geometry: by fixing the pair (x, α), the structure maps describe the linear structure of $(\alpha^\top, x)$, resp. of $(x^\top, \alpha)$. In this way, linear pair geometries can be regarded as algebraic objects whose structure is defined by (one or several) "multiplication maps", just like groups, rings or modules, and just like these they form a *category* (as usual, morphisms are maps that are compatible with the structural data). In Sect. 4 the class of *generalized projective geometries* will be singled out by requiring certain identities for these structure maps.

Example Again in the example of the Grassmann geometry, one can give an explicit formula for the structure maps in terms of linear algebra: identifying elements of $\mathrm{Gras}_E^F(W)$ with images of injective maps $f : E \to W$, modulo equivalence under the general linear group $\mathrm{Gl}_{\mathbb{K}}(E)$ ($f \sim f'$ iff $\exists g \in \mathrm{Gl}_{\mathbb{K}}(E)$: $f' = f \circ g$), and elements of $\mathcal{X}^- = \mathrm{Gras}_F^E(W)$ with kernels of surjective maps $\phi : W \to E$, again modulo equivalence under $\mathrm{Gl}_{\mathbb{K}}(E)$, the basic transversality relation is: $[f]\top[\phi]$, if and only if $\phi \circ f : E \to E$ is a bijection. Then the structure map Π_r is given by the explicit formula

$$\Pi_r([f], [\phi], [h]) = \big[(1 - r) f \circ (\phi \circ f)^{-1} + rh \circ (\phi \circ h)^{-1}\big].$$

(Proof, cf. [8]: first of all, note that the expression on the right-hand side is independent of the chosen representatives; then choose new representatives such that $\phi \circ f = \mathrm{id}_E = \phi \circ h$, and observe that this gives the usual formula of a barycenter in an affine space.) As in ordinary projective geometry, the affine picture in the model space $\mathrm{Hom}(E, F)$ is given by writing $f : E \to W = F \oplus E$ as "column vector" and normalizing the second component to be $\mathbf{1}_E$, the identity map of E, and similarly for the "row vector" $\phi : F \oplus E \to E$ (see [8]). To get a feeling for the kind of non-linear formulas that appear in such contexts, the reader may rewrite the preceding formula for Π_r by replacing f, ϕ and h by such column-, resp. row vectors, and then renormalize the right-hand side, in order to get the formula for the multiplication map in the affine picture. The special case $r = \frac{1}{2}$ (the "midpoint map") is particularly important from a Jordan-theoretic point of view.

2.2.3 Base Points

Although this may seem pedantic, we insist in clearly distinguishing between linear pair geometries and those *with base point*: a *base point* is a transversal pair, often denoted by (o^+, o^-) or (o, o'), that is chosen to be fixed "once and for all" (or, at least, until to the end of the present sentence). Whereas geometric concepts should be base point-free, our description of the universe often uses them—from our ant's perspective we often do not realize that our visible world is just a part of the whole universe, and we take this part for the whole. This remark applies to special relativity as well as to quantum mechanics.

Example We explain the last statement. This can be done both in the context of abstract C^*-algebras or in the concrete realization on a Hilbert space $\mathcal{H}$. For simplicity, let us start with the latter (the more abstract setting will be considered in Example 2 of 2.3.2): let $\mathcal{H}$ be a finite or infinite dimensional complex Hilbert space. Our geometry $(\mathcal{X}^+, \mathcal{X}^-)$ will be a subgeometry of the Grassmann geometry $(\mathcal{Y}^+, \mathcal{Y}^-) = (\mathrm{Gras}_{\mathcal{H}}^{\mathcal{H}}(W), \mathrm{Gras}_{\mathcal{H}}^{\mathcal{H}}(W))$ where $W = \mathcal{H} \oplus \mathcal{H}$. Note that here $\mathcal{Y}^+ = \mathcal{Y}^-$ as sets. Now we define $\mathcal{X}^+ = \mathcal{X}^-$ to be the *space of Lagrangian subspaces on W for the Hermitian form* (of "signature (∞, ∞)")

$$\omega : W \times W \to \mathbb{K}, \quad \omega((u, v), (u', v')) = \langle u, v' \rangle + \langle v, u' \rangle.$$

In other words, $(\mathcal{X}^+, \mathcal{X}^-)$ is the subgeometry of $(\mathcal{Y}^+, \mathcal{Y}^-)$ fixed under the involutive automorphism "orthocomplementation w.r.t. ω" (by definition, a Lagrangian subspace E is such that $E = E^\perp$). As in the preceding examples, it is an exercise in linear algebra to show that this affine pair geometry is modelled on the space of Hermitian operators $\mathrm{Herm}(\mathcal{H})$.

The usual framework of quantum mechanics is simply obtained by fixing a base point (o, o') in the Lagrangian geometry $(\mathcal{X}^+, \mathcal{X}^-)$, where o' singles out an affine part $\mathrm{Herm}(\mathcal{H})$ in $\mathcal{X}^+$ in which o is the zero vector. Of course, $\mathcal{H}$ should then be infinite dimensional. If $\mathcal{H}$ is finite dimensional, say $\mathcal{H} = \mathbb{C}^n$, then $\mathcal{X}^+$ and $\mathcal{X}^-$ are homogeneous spaces under the action of the projective pseudo-unitary group $\mathbb{P}\mathrm{U}(n, n)$, with stabilizer P of o being a certain maximal parabolic subgroup, so that $\mathcal{X}^+ \cong \mathbb{P}\mathrm{U}(n, n)/P$ is modelled on the (Jordan algebra of) Hermitian $n \times n$-matrices $\mathrm{Herm}(n, \mathbb{C})$. Now, for $n = 2$, this Jordan algebra is isomorphic to Minkowsi space $\mathbb{R}^{(3,1)}$, the group $\mathbb{P}\mathrm{U}(2, 2)$ is isomorphic to the conformal group $\mathrm{SO}(4, 2)$ of Minkowski space, and $\mathcal{X}^+ \cong \mathrm{SO}(4, 2)/P$ is precisely the *conformal compactification of Minkowski space*. Therefore our geometric setting of quantum mechanics can be seen as the infinite dimensional analog of the geometric, conformal completion of usual, flat special relativity. The fact that quantum mechanics and special relativity appear as *linear* theories corresponds to the fact that a base point (o, o') has been fixed.

2.3 Energy

As long as the two dual worlds $\mathcal{X}^+$ and $\mathcal{X}^-$ remain neatly separated, we are in the realm of "projective geometry", which is a beautiful theory, but lacks the dynamics that we are used to from the real world. In order to create dynamics, we must introduce some sort of identification between $\mathcal{X}^+$ and $\mathcal{X}^-$ which, mathematically, is modelized by a pair of bijections $p^+ : \mathcal{X}^+ \to \mathcal{X}^-$, $p^- : \mathcal{X}^- \to \mathcal{X}^+$. Henceforth, the observable x and the observer $p^+(x)$ will be considered as "the same thing", and similarly the observer α and the observable $p^-(\alpha)$ are the same thing. Consistency requires then that p^- is the inverse transformation of p^+, and moreover that (p^+, p^-) respects the laws mentioned above (i.e., it defines an isomorphism of the geometry onto its dual geometry). Geometrically, this corresponds to what is sometimes called a *correlation* in projective geometry, but from the point of view of physics, it seems that it really is some sort of *energy*. Since it acts as a transformation, one might be tempted to use also the term *Hamilton operator* for p^+ or for its inverse p^-.

2.3.1 Polarities

There is something special about human beings, namely that *we can observe ourselves*. Let us call an observer *active* or *non-isotropic* if it has this property, i.e., if $\alpha \top p^-(\alpha)$, and *passive* or *isotropic* else. In order to create dynamics, we must require that at least some active observers shall exist; then our correlation (p^+, p^-) is called a *polarity*. In the remainder, we will mainly be concerned with the "active universe" $\mathcal{M}^{(p)} = \{x \in \mathcal{X}^+ | \, x \top p^+(x)\}$. We do not require that the whole universe is active—this may happen for very strong energies which we call *elliptic polarities*, but in general, it seems that polarities of hyperbolic type are more interesting since singular points, such as possible "beginnings" or "ends of the universe", will have to be passive.

Example Polarities of projective spaces are constantly used in classical geometry: assume W is a real Hilbert space; then one identifies a line $[x] \in \mathcal{X}^+ = \mathbb{P}(W)$ with its orthogonal hyperplane $[x]^\perp \in \mathcal{X}^- = \mathbb{P}(W^*)$. Since a scalar product is positive, there are no isotropic vectors: the polarity is elliptic. But we may also work with general non-degenerate forms (symmetric or skew-symmetric) and then get more general polarities. If we work over complex Hilbert spaces, then the scalar product induces a $\mathbb{C}$-antilinear polarity (so we are working with complex geometries, considered as real ones); there are of course also $\mathbb{C}$-linear polarities, coming from non-degenerate $\mathbb{C}$-bilinear forms on W, but their polarities always have isotropic elements. It is clear that such kinds of "orthocomplementation polarities" can be defined also for Grassmann geometries (provided that the $\mathbb{K}$-module W admits suitable non-degenerate bi- or sesquilinear forms). In any case, the affine picture of such kinds of polarities is given by identifying a linear operator from $\mathrm{Hom}(E, F)$ with a suitable adjoint in $\mathrm{Hom}(F, E)$.

2.3.2 Null-Systems

On the other hand, it is theoretically possible that energies are so weak that they admit no active observables whatsoever; one would call them *null-energies* or *null-systems*. It is even theoretically conceivable that there be an *absolute null-energy* which is defined to be a pair (n^+, n^-) of mutually inverse bijections as above, commuting with *all* internal symmetries of the universe $(\mathcal{X}^+, \mathcal{X}^-)$; we then say that our geometry is *of the first kind*, or a *null geometry*. In this case, the identification $\mathcal{X}^+ \cong \mathcal{X}^-$ is much more canonical than for any Hamilton operator, so that it makes indeed sense to call the point $n^-(\alpha)$ for an observer α its "point at infinity". An apparently trivial example of this situation is the projective line $\mathcal{X}^+ = \mathbb{KP}^1$, which is canonically the same as its dual projective line $\mathcal{X}^-$ of "hyperplanes" (=points) in $\mathbb{KP}^1$; but clearly this identification is a null-system (a line is never transversal to itself!) and not a polarity (algebraically, this identification comes from the canonical symplectic form on $\mathbb{K}^2$). We are thus forced to switch constantly between two *different* ways of identifying $\mathcal{X}^+$ and $\mathcal{X}^-$, namely by the Hamilton operator p, who governs the geometry of the active world, and by the "underlying" null energy n. This is indeed a good reason for distinguishing $\mathcal{X}^+$ and $\mathcal{X}^-$ from the outset. Without this clear distinction, one arrives at paradoxes such as "every active observer is both identical with its zero vector and with its point at infinity". The observer might simply say "I am my origin and my point at infinity".

Example 1 The projective line $\mathrm{Gras}_1(\mathbb{K}^2)$ is generalized by the Grassmannians $\mathrm{Gras}_n(\mathbb{K}^{2n})$, or, in arbitrary dimension, by the "type = cotype" Grassmann geometry $(\mathrm{Gras}_E^E(W), \mathrm{Gras}_E^E(W))$ with $W = E \oplus E$: the identity map $\mathcal{X}^+ \to \mathcal{X}^-$ is indeed a canonical null-system.

Note that in this case $\mathcal{X}^+$ and $\mathcal{X}^-$ are modelled on $\mathrm{End}(E) = \mathrm{Hom}(E, E)$, which is an *associative algebra*.

The Lagrangian geometry introduced above is also a null geometry, where the null system is the identity map from $\mathcal{X}^+$ to $\mathcal{X}^-$. Since quantum mechanics corresponds to the choice of a base point (o, o') in this geometry, polarities should always be compatible with this base point, i.e., $o' = p^+(o)$. The fixed Hilbert structure on $\mathcal{H}$ corresponds to the choice of the *standard elliptic polarity* given by orthocomplementation in the Hilbert space $W = \mathcal{H} \oplus \mathcal{H}$, or to the *standard hyperbolic polarity*, given by orthocomplementation with respect to the neutral form $\beta((u, v), (u', v')) := \langle u, u' \rangle - \langle v, v' \rangle$.

Example 2 The example of the projective line is indeed quite typical: it is not misleading to consider "null geometries" as a rather subtle generalisation of the projective line. In fact, as mentioned above, $\mathrm{End}(E)$ is an associative algebra; so let us start with a general associative algebra A and define $\mathcal{X}^+ = \mathcal{X}^-$ to be the *projective line over* A, which by definition (cf., e.g., [17]) is the set $A\mathbb{P}^1 := \mathrm{Gras}_A^A(A \oplus A)$ of all submodules of the left A-module $A \oplus A$ that are isomorphic to A and admit a complementary submodule isomorphic to A. This geometry is modelled on A, and again the identity map $\mathcal{X}^+ \to \mathcal{X}^-$ is a canonical null-system.

More important for physics are subgeometries of the projective line that are induced by fixing an *involution* $* : A \to A$ (antiautomorphism of order 2; if the base field $\mathbb{K}$ itself carries a distinguished involution, we may also require that $*$ is antilinear with respect to this involution). Then the involution $*$ lifts to an involution of the projective line $A\mathbb{P}^1$ whose fixed point set is called the *Hermitian projective line*, cf. [14], Sect. 8. Again, this is a null geometry, and it generalizes the Lagrangian geometry from Example 1.2.3. It is therefore the geometric object corresponding to the abstract C^*-algebra approach to quantum mechanics.

Of course, quantum mechanics requires to work over the field $\mathbb{C}$ of complex numbers and the involution $*$ to be $\mathbb{C}$-antilinear. This has the particular consequence that the spaces of Hermitian elements ($a^* = a$) and skew-Hermitian elements ($a^* = -a$) are isomorphic, whereas for more general involutions this need not be the case (e.g., real square matrices with $*$ being the usual transpose). Correspondingly, in the general case there is also a *skew-Hermitian projective line*, which in general is not isomorphic to the Hermitian one (to be a bit more precise: the skew-Hermitian projective line essentially corresponds to the unitary group of $(A, *)$, considered as a homogeneous space under an even bigger group), but in the case of quantum mechanics happens to be isomorphic to the Hermitian projective line. This "accident" corresponds to the ambiguity in the interpretation of observables (see the quotation from [1] given in the Sect. 1), and we have the impression that it is also related to the problem of time development ("unitary **U**-evolution versus state reduction **R**"; see item (4) in the Sect.1 and Sect. 2.3.3.2 below).

2.3.3 Dynamics

We affirmed above that a polarity creates dynamics. This needs explanation: as a general fact, any polarity of a generalized projective geometry defines, on the "active universe" $\mathcal{M}^{(p)}$, the structure of a *symmetric space*—thus there is a canonical torsion free *affine connection* together with its associated groups, and the notions of *geodesics* and of *geodesic flow* on the tangent bundle $T\mathcal{M}^{(p)}$ are defined. This is best seen by looking at some examples, at least in the finite-dimensional real case; in the general infinite-dimensional case, these notions are somewhat less standard, and we comment on this below.

Example Let us first look at elliptic polarities of finite-dimensional geometries, which lead to *compact symmetric spaces*: in all cases, this polarity is given by orthocomplementation with respect to a (real or complex) scalar product, so the relevant automorphism groups are orthogonal, resp. unitary groups, which act transitively on the geometry. We thus can write $\mathcal{X}^+ \cong U/K$, U a compact Lie group and K essentially the group of fixed points of an involution σ of U:

- for the real Grassmannians $\mathcal{X}^+ \cong \mathrm{O}(p+q)/\mathrm{O}(p) \times \mathrm{O}(q)$,
- for the complex Grassmannians $\mathcal{X}^+ \cong \mathrm{U}(p+q)/\mathrm{U}(p) \times \mathrm{U}(q)$ (the special case $p = 1$ corresponds to ordinary projective spaces),
- for the complex Hermitian Lagrangian geometry $\mathcal{X}^+ \cong \mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{diag} \cong \mathrm{U}(n)$.

These spaces are well known to be compact symmetric spaces, and the last two series are moreover *Hermitian symmetric spaces* [23, 31]. It is known that all compact symmetric spaces of "classical type" can be obtained in a similar way (cf. [5]). The *non-compact dual* G/K of U/K can be obtained by taking a slightly modified polarity (hyperbolic polarity); in this case, the active universe $M^{(p)}$ is a dense open subset of $\mathcal{X}^+$ which is no longer connected, but one of the connected components is always the non-compact dual G/K (the inclusion $G/K \subset \mathcal{X}^+ = U/K$ generalizes the well-known *Borel-imbedding*, cf. [23]). For instance, the non-compact duals of the projective spaces are the *real, resp. complex hyperbolic spaces*, realized as "balls" in $\mathbb{R}^n$, resp. $\mathbb{C}^n$ (cf. also [22]). Finally, it is also possible to choose polarities that are of yet different type, such that $\mathcal{M}^{(p)}$ is a *pseudo-Riemannian* symmetric space. For instance, the de Sitter and anti-de Sitter models of general relativity are obtained by suitable polarities of the Lagrangian geometry for $n = 2$. Replacing $\mathbb{C}^n$ by a Hilbert space and the groups $\mathrm{U}(p, q)$ by suitable infinite dimensional unitary groups, these examples generalize. In particular, the elliptic polarity of the infinite dimensional Lagrangian geometry leads to the identification of the "conformal completion $\mathcal{X}^+$ of $\mathrm{Herm}(\mathcal{H})$" with the unitary group $\mathrm{U}(\mathcal{H})$, which here is seen as an infinite dimensional Hermitian symmetric space $\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})/\mathrm{diag} \cong \mathrm{U}(\mathcal{H})$.

Concerning the dynamical system induced by a polarity, there are (at least) two serious problems:

2.3.3.1 Integration The first problem is of analytic nature. It arises when we try to "integrate" the differential equations of the dynamical system, for instance, to obtain the geodesic flow corresponding to a spray. Here, properties of the base ring $\mathbb{K}$ start to play a rôle: whereas the choice of $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ implies a tendency towards a more deterministic behavior (via existence and uniqueness theorems for ordinary differential equations), this may be much less the case for other base fields or rings (for instance, there are no general existence and uniqueness theorems for p-adic differential equations). For physics this may seem not to be a too serious issue since one is mainly interested in real or complex Hilbert or Banach space settings and not in "too wild" infinite dimensional situations where basic theorems on ordinary differential equations may fail.

2.3.3.2 The **U***-evolution* The second problem is of a more geometric nature. It arises when we try to identify, in the case of the Lagrangian geometry corresponding to usual quantum mechanics, the **U**-*evolution* (unitary evolution defined by the Schrödinger equation) with a flow defined in terms of geometry (say, with the geodesic flow of a symmetric space). If a base point (o, o'), an additional observable H and a polarity p are fixed, there are several possibilities to associate dynamical systems to this situation, but at present none of them really seems to coincide with the usual **U**-evolution. Perhaps should it be necessary to use the description by an observer $\alpha = \alpha(t)$ which also moves in $\mathcal{X}^-$, in order to obtain the usual picture? Or is it indeed necessary to invoke some "associative feature" that cannot be captured by a Jordan description alone (cf. the quotation from [1])? In [2] it is explained, following Kibble [30], that the Schrödinger evolution can indeed be interpreted as a Hamiltonian flow in a suitable geometric setting; thus, at least, it seems reasonable to look for an interpretation of the **U**-evolution in geometric terms also in the present setting.

2.3.4 Curvature

Regardless whether we are able to integrate differential equations or not, there is always a notion of *curvature* of an affine connection—in general, the symmetric spaces defined by a polarity will have non-vanishing curvature. The curvature tensor is a trilinear map satisfying the defining identities of a *Lie triple system* (see Sect. 3.6), and it contains all local information about the symmetric space, just as does the Lie algebra of a Lie group. The Jordan product is closely related to the Lie triple system (Sect. 3.6), and thus the Jordan product can geometrically be interpreted as a curvature feature.

2.4 States and Pure States

We started with observables (and observers) as fundamental objects of our theory, and not with *states*. On the other hand, in usual quantum mechanics, the pure states form a projective space $\mathbb{P}(\mathcal{H})$, and it might seem more natural to take the "geometry of pure states" $(\mathbb{P}(\mathcal{H}), \mathbb{P}(\mathcal{H}^*))$ as basic object of a geometric approach to quantum mechanics—this is indeed the common ground of all such approaches we know about, see, e.g., [2, 16, 19, 30, 42]. What is the relation between these approaches and the one proposed here?

In a linear pair geometry $(\mathcal{X}^+, \mathcal{X}^-)$ there is a natural notion of *intrinsic subspace* or *state (in $\mathcal{X}^+$)*: it is defined as a subset $\mathcal{I} \subset \mathcal{X}^+$ which, to *any* observer $\alpha \in \mathcal{X}^-$, appears linearly, i.e., $\mathcal{I} \cap \alpha^\top$ is a linear subspace of $\alpha^\top$, regardless which origin $o \in \mathcal{I} \cap \alpha^\top$ we choose, and it is called *minimal* or a *pure state* if it is of *rank* 1, i.e., it is not reduced to a point and does not properly include intrinsic subspaces that are not points. Similarly, *states in $\mathcal{X}^-$* are defined; we may call them "dual states" or "kets". The best way to get some idea on these concepts is to look at examples:

Example 1 States of the Grassmann geometry $(\mathcal{X}^+, \mathcal{X}^-) = (\mathrm{Gras}_E^F(W), \mathrm{Gras}_F^E(W))$ can be constructed as follows: fix some flag $0 \subset F_1 \subset F_2 \subset W$ and let

$$\mathcal{I} := \{A \in \mathcal{X}^+ \mid F_1 \subset A \subset F_2\}$$

be the set of all elements of the Grassmannian that are "squeezed" by this flag. Then $\mathcal{I}$ is an intrinsic subspace. Conversely, if $\mathbb{K}$ is a field and $W = \mathbb{K}^n$ is finite-dimensional, then all intrinsic subspaces are of this form ([12], Theorem 3.11). Such a state is pure if the codimensions are minimal, i.e., if $\dim E = \dim F_1 + 1 = \dim F_2 - 1$. In particular, in the case of an ordinary projective geometry $\mathbb{P}(W)$ (i.e., $\dim E = 1$), states are the usual projective subspaces in $\mathbb{P}(W)$, and pure states are projective lines in $\mathbb{P}(W)$. As is well-known, every affine subspace in the affine picture then corresponds to a state (namely, to the projective subspace which is its completion). The situation changes drastically if $\dim E > 1$: then only rather specific affine subspaces of the affine part belong to states, namely the so-called *inner ideals* of the corresponding Jordan pair (see Sect. 3.7). In particular, it follows that pure states through the origin of the affine part $\mathrm{Hom}(E, F)$ are represented by rank-one operators in the usual sense.

Example 2 States of the Lagrangian geometry are constructed similarly, by taking *Lagrangian flags* (i.e., flags $0 \subset F_1 \subset F_2 \subset W$ with $F_1^{\perp} = F_2$). Then similar results as in the Grassmann case hold; in particular, pure stats through the origin of the linear part $\mathrm{Herm}(\mathcal{H})$ are represented by Hermitian rank-one operators, i.e., by projectors on one-dimensional subspaces of $\mathcal{H}$. We thus recover the space of pure states of usual quantum mechanics: it is the space of all minimal intrinsic subspaces running through a fixed "origin" $o \in \mathcal{X}^+$.

2.4.1 The Geometry of States

We denote by $\mathcal{S}^{\pm}$ the collection of all states in $\mathcal{X}^{\pm}$, or perhaps of all pure states, or of states of some given rank r, and we think of states as elements of a new geometry $(\mathcal{S}^+, \mathcal{S}^-)$ which is associated to our universe $(\mathcal{X}^+, \mathcal{X}^-)$ in a similar way as, for example, one associates to a usual pair of dual projective spaces $(\mathbb{P}V, \mathbb{P}V^*)$ the geometry of *all* projective subspaces, or of subspaces of a given rank. One may ask whether $(\mathcal{S}^+, \mathcal{S}^-)$ is again a "good" geometry—a look at our standard examples shows that this is indeed quite often the case:

Example The preceding Example 1 shows that intrinsic subspaces of Grassmannians correspond to *short flags* $0 \subset F_1 \subset F_2 \subset W$, where the rank of the intrinsic subspace corresponds to the *type of the flag* (characteristic dimensions, for instance). Thus $(\mathcal{S}^+, \mathcal{S}^-)$ is a flag geometry. As mentioned in the example of Sect. 2.2, such geometries are again linear pair geometries. However, they are in general no longer affine pair geometries: indeed, they are of the form $(G/P^-, G/P^+)$, where the parabolic subgroups $P^{\pm}$ are no longer associated to 3-gradings, but to 5-gradings (see Sect. 3.1). Such geometries attract much interest in current research since they are related to "non-commutative Jordan structures" and also to exceptional geometries. The same remarks hold for the geometry of states of a Lagrangian geometry: it corresponds to certain 5-gradings of the Lie algebra of $\mathrm{SU}(n, n)$.

Coming back to the geometry of quantum mechanics, we now have to carefully distinguish between two notions of "pure states": first, the usual one (minimal intrinsic subspaces running through the fixed origin, leading to projective space $\mathbb{P}(\mathcal{H})$), and second, the new base point-free interpretation (leading to the geometry of short Lagrangian flags). In view of the preceding remarks, this change corresponds to passing from usual, commutative Jordan algebraic structures to non-commutative ones. Thus we enter into a "non-commutative Jordan geometry"—if the term "second quantization" were not already taken, it would be tempting to use it here. Of course, we do not know if this new non-commutativity is reflected anywhere in physical reality, but at least theoretically there might be some possibility here to test experimentally whether the second interpretation has some physical meaning.

2.4.2 Comparison with Twistor Geometry

Another striking feature of our interpretation of states is the similarity with the double fibrations from Penrose's twistor theory: the interpretation of states as intrinsic subspaces produces a new duality—a duality between the "geometry of observables" $(\mathcal{X}^+, \mathcal{X}^-)$ and the "geometry of states" $(\mathcal{S}^+, \mathcal{S}^-)$. This new duality corresponds to the double fibration

$$\begin{array}{ccc} & \mathcal{Z}^+ & \\ \swarrow & & \searrow \\ \mathcal{X}^+ & & \mathcal{S}^+ \end{array}$$

where $\mathcal{Z}^+ = \{(x, \mathcal{I}) | x \in \mathcal{I}\} \subset \mathcal{X}^+ \times \mathcal{S}^+$ is the "incidence space" ("x is incident with $\mathcal{I}$").

Example For $\mathbb{K} = \mathbb{C}$ and $\mathcal{X}^+ = \mathrm{Gras}_2(\mathbb{C}^4)$, $\mathcal{S}^+ = \mathbb{CP}^3 = \mathrm{Gras}_1(\mathbb{C}^4)$, we get the complexified setting of Penrose's twistor theory as described in [3], p. 8. Here, $\mathcal{S}^+$ rather corresponds to a geometry of *maximal* intrinsic subspaces. In the more sophisticated setting from [38], Chap. 33, starting with (compactified) *real* Minkowski space, light rays indeed correspond to minimal intrinsic subspaces, and $\mathcal{S}^+$ forms a 5-graded geometry of Lagrangian flags associated to $\mathrm{SU}(2,2)$. In any case, our setting incorporates aspects of *non-locality* that have been a main motivation of Penrose's for developing twistor theory: a pure state is a "line" and thus is a *global* object of our universe $\mathcal{X}^+$. Note finally the change of rôle of "observables" and "states": by analogy with quantum theory, we are driven to call "pure state" the light rays and not the points of Minkowski space, which rather correspond to "observables", whereas classically points of a manifold are rather viewed as pure states. Compare with [38], p. 964: "there is a striking reversal of this in twistor space, since now the light ray is described as a *point* and an event is described as a *locus*."

2.5 Open Ends

The reader who wishes to recover the familiar picture of quantum mechanics will wonder (at least) about the following questions:

(1) How is "measurement" and "state reduction" described? As to the mathematical framework, the analog of "eigenvectors" and "diagonalization" (with respect to some complete family of pure states through a given point, which is the analog of a Hilbert basis) is perfectly well-defined (I will not go into details; the specialist may look at [33]). As to its interpretation as "state reduction", I prefer not to make any statements—following the advice of J. Bell ([4], p. 126): "Concepts of 'measurement', or 'observation', or 'experiment' should not appear at a fundamental level."
(2) What, after all, shall be the interpretation of the "universe" $(\mathcal{X}^+, \mathcal{X}^-)$: shall we think of it as space-time of relativity, or quantum mechanically as a *single* particle, or as a *many-particle system*, or as a field of "all" particles? According to the standard formalism of quantum mechanics, there should be some way to compose a many-particle system from single particles, formalized by the tensor product of Hilbert spaces; but there is no "tensor product of Jordan algebras", and accordingly there is no obvious notion of "tensor product of generalized projective geometries". This is indeed a serious problem; I will comment in the last section on some possible ways of attack.

3 Jordan Theory

3.1 Historical Remark

As mentioned in Sect. 1, Pascual Jordan's Ansatz started with the observation that any associative algebra, such as finite or infinite-dimensional matrix algebras, equipped with the new product $x \bullet y = \frac{1}{2}(xy + yx)$, becomes a commutative algebra which is not associative but satisfies another identity, namely

$$(x^2 \bullet y) \bullet x = x^2 \bullet (y \bullet x).$$

In case that 2 is invertible in $\mathbb{K}$ (which we will always assume), this identity then served as axiomatic definition of a class of commutative algebras, the later so-called *Jordan algebras* (see Part I of [34] for a detailed historical survey of Jordan theory). Unfortunately, this axiomatic definition is much less appealing than the one of a Lie algebra, and therefore we prefer to skip some 40 years of historical development and turn right away to more general objects that are easier understood, namely to *Jordan pairs* and *Jordan triple systems*. Let us, however, point out that a Jordan algebra is called *special* if it is a subalgebra of some associative algebra with the symmetrized bullet-product; there exists essentially only one exceptional Jordan algebra (the 27-dimensional *Albert algebra*, which has attracted some attention also in physics, most notably by work of M. Günaydin and his collaborators, see the extensive bibliography in [24]). The other classical Jordan algebras are:

Example Here are the main families of special Jordan algebras:

(1) full matrix algebras $M(n,n;\mathbb{K})$ (full endomorphism algebras of a linear space),
(2) symmetric and Hermitian matrices $\mathrm{Sym}(n,\mathbb{K})$, resp. $\mathrm{Herm}(n,\mathbb{K})$ (selfadjoint operators of an inner product space),
(3) skew-symmetric matrices in even dimension (selfadjoint operators with respect to a symplectic form),
(4) spin factors: vector spaces V with symmetric bilinear form $\beta : V \times V \to \mathbb{K}$ and a distinguished element e with $\beta(e,e) = 1$, with product

$$x \bullet y := \beta(x,e)y + \beta(y,e)x - \beta(x,y)e.$$

3.2 Jordan Pairs and 3-Graded Lie Algebras

A $2k+1$-*graded Lie algebra* is a Lie algebra of the form

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

($k \in \mathbb{N}$) such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. By convention, $\mathfrak{g}_i = 0$ for $i \notin \{-k,\dots,k\}$. We are particularly interested in 3-graded Lie algebras ($k = 1$); then our the condition implies that $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are Abelian subalgebras of $\mathfrak{g}$. Let $V^\pm := \mathfrak{g}_{\pm 1}$. We define trilinear products

$$T^\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm, \quad (x,y,z) \mapsto [[x,y],z].$$

From the Jacobi identity in $\mathfrak{g}$, together with the fact that $\mathfrak{g}_{\pm 1}$ are Abelian subalgebras of $\mathfrak{g}$, we then easily get the following two identities:

$$T^\pm(x,y,z) = T^\pm(z,y,x), \tag{LJP1}$$

$$\begin{aligned} &T^\pm(a,b,T^\pm(x,y,z)) \\ &\quad = T^\pm(T^\pm(a,b,x),y,z) - T^\pm(x,T^\mp(b,a,y,z)) + T^\pm(x,y,T^\pm(a,b,z)). \end{aligned} \tag{LJP2}$$

By definition, a (*linear*) *Jordan pair* is a pair (V^+, V^-) of $\mathbb{K}$-modules together with trilinear maps $T^\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm$ satisfying (LJP1) and (LJP2). Every linear Jordan pair is obtained by the construction just described (sometimes this is called the *Kantor-Koecher-Tits construction*); the easiest proof is by noting that $V^+ \oplus V^-$ carries the structure of a (polarized) *Lie triple system*, and then the so-called *standard imbedding* of this Lie triple system yields a 3-graded Lie algebra (cf. [5], Chap. III).

Example All classical Lie algebras admit 3-gradings (if we allow $\mathfrak{sl}(n,\mathbb{K})$ to be replaced by $\mathfrak{gl}(n,\mathbb{K})$). Let us consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}(W)$ of all endomorphisms of a linear space

W and fix a direct sum decomposition $W = E \oplus F$. Then we have an associated 3-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_0$, where, in an obvious matrix notation,

$$\mathfrak{g}_1 := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathrm{Hom}(F, E) \right\}, \qquad \mathfrak{g}_{-1} := \left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \mid y \in \mathrm{Hom}(F, E) \right\},$$

and $\mathfrak{g}_0$ given by the "diagonal matrices" in $\mathfrak{g}$. Calculating the triple bracket $[[x, y], z]$ for $x, z \in \mathfrak{g}_1$, $y \in \mathfrak{g}_{-1}$, we see that the corresponding Jordan pair is

$$(V^+, V^-) = (\mathrm{Hom}(F, E), \mathrm{Hom}(E, F)) \quad \text{with} \quad T^{\pm}(x, y, z) = xyz + zyx.$$

Next, the symplectic Lie algebra $\mathfrak{sp}(\mathcal{H} \oplus \mathcal{H})$ and the orthogonal Lie algebra $\mathfrak{o}(\mathcal{H} \ominus \mathcal{H})$, with respect to the symplectic form $\omega((u, v), (u', v')) = \langle u, v' \rangle - \langle v, u' \rangle$, resp. with respect to the symmetric form $\beta((u, v), (u', v')) = \langle u, v' \rangle + \langle v, u' \rangle$ on $\mathcal{H} \oplus \mathcal{H}$, are subalgebras of $\mathfrak{gl}(\mathcal{H} \oplus \mathcal{H})$ and inherit from it a 3-grading. The corresponding Jordan pairs are of the form $(V^+, V^-) = (V, V)$ where V is the space of symmetric (or Hermitian), resp. of skew-symmetric (or skew-Hermitian) matrices, with trilinear map given by the same formula as above. The similarity in notation with the description of Grassmann and Lagrangian geometries is of course not an accident (see explanations below). Finally, some orthogonal Lie algebras admit another family of 3-gradings, leading to Jordan pairs related to the spin-factors defined above. Exceptional Lie algebras not always admit a 3-grading; for instance, G_2, F_4 and E_8 do not, whereas some forms of E_6 and E_7 do.

3.3 Jordan Triple Systems and Involutive 3-Graded Lie Algebras

Assume now that $\mathfrak{g}$ is a 3-graded Lie algebra together with an involution θ, i.e., an automorphism of order 2 reversing the grading: $\theta(\mathfrak{g}_i) = \mathfrak{g}_{-i}$. Then let $V := V^+ = \theta(V^-)$ be equipped with the trilinear product

$$T(x, y, z) := T^+(x, \theta y, z) = [[x, \theta y], z]$$

which clearly satisfies the two identities (JT1) and (JT2) obtained from (LJP1) and (LJP2) by omitting the superscripts ± 1. By definition, a *Jordan triple system* is a $\mathbb{K}$-module V together with a trilinear map $T : V \times V \times V \to V$ satisfying (JT1) and (JT2). Every Jordan triple system is obtained by the construction just described: from (V, T) one recovers a Jordan pair $(V^+, V^-, T^{\pm})$ by letting $V^+ := V^- := V$ and $T^{\pm} := T$, and then applies the "Kantor-Koecher-Tits construction" outlined above.

Example All 3-graded Lie algebras from the preceding example admit involutions, but in general there is no distinguished one. The matrix realization given above privileges the involution given by $\theta(X) = -X^t$ (negative transpose), which leads to the matrix triple systems given by $T(X, Y, Z) = -(XY^tZ + ZY^tX)$.

3.4 Unital Jordan Algebras and Invertible Elements

Let $\mathfrak{g}$ be a 3-graded Lie algebra, with corresponding Jordan pair (V^+, V^-). For $x \in V^{\pm}$, the *quadratic operator* is defined by

$$Q^{\pm}(x) : V^{\mp} \to V^{\pm}, \quad y \mapsto \frac{1}{2} T^{\pm}(x, y, x) = \frac{1}{2}[[x, y], x].$$

An element $x \in V^-$ is called *invertible* if the linear map $Q^-(x) : V^+ \to V^-$ is invertible. It can be shown that then $V := V^+$ with the bilinear product

$$y \bullet z := \frac{1}{2} T^+(y, x, z)$$

becomes a Jordan algebra with unit element $e := Q(x)^{-1}(x)$, and that every unital Jordan algebra is obtained in this way. In other words, unital Jordan algebras are the same as Jordan pairs together with a distinguished invertible element.

Example For $\mathfrak{gl}(W)$ with $W = E \oplus F$, the quadratic operator is given by

$$Q^+(x) : \mathrm{Hom}(E, F) \to \mathrm{Hom}(F, E), \quad y \mapsto xyx.$$

If $\mathbb{K}$ is a field and E and F are non-isomorphic vector spaces, then this map cannot be an isomorphism, and hence the Jordan pair never has invertible elements. On the other hand, if E and F are isomorphic as vector spaces, then $Q^+(x)$ is bijective if and only if x is invertible as a linear map from E to F, and then $Q^+(x)^{-1} = Q^-(x^{-1})$. Fixing for a moment such an element x as an identification of E and F, the corresponding Jordan algebra structure on $V := V^+ \cong V^- \cong \mathrm{End}(E)$ is just the usual symmetrized product. Similarly, the Jordan pairs of symmetric and Hermitian matrices always have invertible elements, whereas for $n \times n$-skew symmetric matrices this is true only when n is even.

Note that the bullet-product defined above depends on x. In the case of full matrix pairs $(\mathrm{End}(E), \mathrm{End}(E))$, this dependence is not very serious: as long as x is invertible, all these products are isomorphic, but for Hermitian matrix pairs $(\mathrm{Herm}(\mathcal{H}), \mathrm{Herm}(\mathcal{H}))$, it becomes more subtle: it depends on the isomorphism class of x, seen as a Hermitian form on $\mathcal{H}$; thus in general the various Jordan algebras obtained by fixing the invertible element x need no longer be isomorphic among each other (they are only "isotopic").

3.5 The Fundamental Formula

The following identity which is valid in all Jordan pairs (V^+, V^-) is called the *fundamental formula*: for all $x \in V^-$, $y \in V^+$,

$$Q^-(Q^-(x)y) = Q^-(x)Q^+(y)Q^-(x).$$

There seems to be no "straightforward proof" of this formula, starting from the definition of a linear Jordan pair given in Sect. 3.2. In [32] this formula is taken as one of the *defining* axioms of a Jordan pair.

Example In case of the Jordan pair of rectangular matrices, $(\mathrm{Hom}(E, F), \mathrm{Hom}(F, E))$, the proof of the fundamental formula is easy: as seen above $Q(x)z = xzx$, and hence $Q(Q(x)y)z = Q(xyx)z = xyxzxyx = Q(x)Q(y)Q(x)z$.

3.6 The Jordan-Lie Functor

To every Jordan triple system (V, T) we associate a new ternary product $R = R_T$ by anti-symmetrizing in the first two variables:

$$[X, Y, Z] := R_T(X, Y)Z := T(X, Y, Z) - T(Y, X, Z).$$

It is easily shown that this trilinear product is a *Lie triple system*, i.e.,

- it is antisymmetric in X and Y,
- it satisfies the Jacobi identity $[X, Y, Z] + [Y, Z, X] + [Z, X, Y] = 0$,
- the endomorphism $R_T(X, Y) : V \to V$ is a *derivation* of the triple product $[\cdot, \cdot, \cdot]$.

The correspondence $T \mapsto R_T$ is called the *Jordan-Lie functor* (see [5]). Lie triple systems are for symmetric spaces what Lie algebras are for Lie groups: an infinitesimal version that, locally, determines them completely. Moreover, the Lie triple product is (possibly up to a sign) the *curvature tensor of the canonical connection* (see [9, 31]). Many, but not all Lie triple systems are obtained from Jordan triple systems via the Jordan-Lie functor; and some Lie triple systems are obtained from 2 or even 3 different Jordan triple systems (cf. tables in [5], Chap. XII).

Example Consider the Jordan triple system $V^+ = V^- = \mathrm{End}(E)$ with $T(x, y, z) = xyz + zyx$. In this case

$$R_T(x, y)z = xyz + zyx - yxz - xxy = [x, y]z - z[x, y] = [[x, y], z]$$

is the triple Lie bracket in $\mathfrak{gl}(E)$. For other classical Lie algebras, the triple Lie bracket can be similarly obtained from a Jordan triple system, but never for exceptional Lie algebras. "Classical" symmetric spaces essentially can all be realized as subspaces fixed under one or several involutions (involutive automorphisms or anti-automorphisms) of $\mathrm{Gl}(n, \mathbb{K})$, and hence their Lie triple system comes from a Jordan sub-triple system of $M(n, n; \mathbb{K})$ (see [5]); for exceptional symmetric spaces the situation is more difficult.

3.7 Inner Ideals

Left and right ideals of associative algebras are generalized by *inner ideals* in Jordan theory: an *inner ideal* I in the $+$-part V^+ of a Jordan pair (V^+, V^-) is a linear subspace $I \subset V^+$ which is stable under "multiplication from the inside":

$$T^+(I, V^-, I) \subset I.$$

John R. Faulkner has proposed to use inner ideals as a key ingredient for an incidence geometric approach to Quantum Theory [21].

Example Consider the Jordan triple system $V := V^+ = V^- = \mathrm{End}(E)$ with $T(x, y, z) = xyz + zyx$, and let $L \subset \mathrm{End}(E)$ be a left ideal in the usual sense. Then $T^+(L, V^-, L) \subset LVL + LVL \subset L$; hence L is an inner ideal. Similarly for right ideals R. Clearly, intersections of inner ideals are inner ideals; hence $L \cap R$ is an inner ideal. In finite dimension over a field, all inner ideals of V^+ are of this form—see [12], Appendix A for an elementary account on this.

4 Generalized Projective Geometries

This section makes the link between the preceding two ones: we define *generalized projective geometries* and explain the *Jordan functor*: Jordan structures are for generalized projective (and polar) geometries what Lie algebras are for Lie groups, namely an "infinitesimal version". This correspondence works even better than in the Lie case: we can go backwards and "integrate" Jordan structures to geometries, regardless of the dimension and of the nature of the base field. Summing up, Jordan algebraic structures and their corresponding geometries (equipped with a base point) are completely equivalent. As before, $\mathbb{K}$ is a commutative unital base ring, such that 2 and 3 are invertible in $\mathbb{K}$. (The generalization of the following concepts to the case of characteristic 2 is an interesting open problem.)

4.1 Some Categorial Notions for Linear Pair Geometries

Recall the notion of *linear* and *affine pair geometries* from Sect. 2.2. Although implicitly most has already been said in Sect. 2, let us be more explicit about categorial notions for such pair geometries, such as *morphisms*, *direct products and function spaces*, *connectedness*, *faithfulness*.

4.1.1 Morphisms I

A *homomorphism* between linear pair geometries $(\mathcal{X}^+, \mathcal{X}^-)$ and $(\mathcal{Y}^+, \mathcal{Y}^-)$ is a pair of maps $(g^+, g^-) : (\mathcal{X}^+, \mathcal{X}^-) \to (\mathcal{Y}^+, \mathcal{Y}^-)$ preserving transversality and being compatible with the structure maps in the sense that

$$g^+\Pi_r(x, \alpha, y) = \Pi_r(g^+(x), g^-\alpha, g^+(y)),$$

and dually. This means simply that g^+ induces by restriction a *linear* map from $(\alpha^\top, x)$ to $((g^-(\alpha))^\top, g^+(x))$, and dually. In particular, we can speak of the *automorphism group* $\mathrm{Aut}(\mathcal{X}^+, \mathcal{X}^-)$. If a base point (o^-, o^+) is fixed, then we call *structure group* the group $\mathrm{Aut}(X^+, X^-; o^+, o^-)$ of automorphisms fixing the base point. From the definitions it follows that this group acts linearly on the linear space $(o^-)^\top \times (o^+)^\top$.

4.1.2 Morphisms II

Adjoint or *structural pairs of morphisms* are given by pairs $g : \mathcal{X}^+ \to \mathcal{Y}^+$, $h : \mathcal{Y}^- \to \mathcal{X}^-$ such that transversality is preserved in the sense that $x \top h(\alpha)$ iff $g(x) \top \alpha$, and, whenever $(x, h(\alpha))$ and $(y, h(\alpha))$ are transversal, then

$$g\Pi_r(x, h\alpha, y) = \Pi_r(gx, \alpha, gy),$$

and similarly for Π_r^-, Σ^+ and Σ^-. We write $h = g^t$ if (g, h) is a structural pair. The condition means that g induces a *linear* map from $((h\alpha)^\top, x)$ to $(\alpha^\top, gx)$. Note that every isomorphism (g^+, g^-) in the sense of 4.1.1 gives rise to a structural pair $(g^+, (g^-)^{-1})$, and conversely, every bijective structural pair gives rise to an isomorphism. Thus we have two different categories, but isomorphisms are essentially the same in both of them. For more flexibility, we may consider, in both categories, pairs of maps that are not necessarily defined everywhere.

Example What is your preferred notion of a *homomorphism of projective spaces*? Do you prefer maps $\mathbb{P}(W) \to \mathbb{P}(V)$ that are induced by *injective* linear maps $W \to V$ (hence are defined everywhere), or do you prefer maps induced by arbitrary non-zero linear maps $W \to V$ (hence are not everywhere defined as maps in the usual sense)? In the first case, you prefer to look at projective spaces as members of the Category I (namely, in this category the projective geometry $(\mathbb{P}(W), \mathbb{P}(W^*)$ is a *simple* object, and hence homomorphisms have to be either injective or trivial). In the second case, you prefer to look at them as members of the Category II (namely, in this category, the dual map f^* of an arbitrary non-zero linear map $f : W \to V$ gives the pair $([f], [f]^t) = [f], [f^*]))$.

4.1.3 Direct Products and Function Geometries

If $(\mathcal{X}_i^+, \mathcal{X}_i^-; \top_i)_{i\in I}$ is a family of linear or affine pair geometries, then the *direct product*, with transversality given by

$$(x_i)_{i\in I} \top (\alpha_i)_{i\in I} \quad \text{iff} \quad \forall i \in I : x_i \top \alpha_i$$

is a linear (resp. affine) pair geometry: the new structure maps are simply the direct products of those of $(\mathcal{X}_i^+, \mathcal{X}_i^-)$.

Example A particularly interesting special case is the direct product of a geometry with its dual geometry, $(\mathcal{X}^+ \times \mathcal{X}^-, \mathcal{X}^- \times \mathcal{X}^+)$. It carries a canonical polarity, namely the *exchange map* $\tau((x,\alpha)) = (\alpha, x)$.

Another important case are *function geometries*: the index set is some geometric space M and all $(\mathcal{X}_i^+, \mathcal{X}^-)$, $i \in M$, are copies of a fixed geometry $(\mathcal{X}^+, \mathcal{X}^-)$. In other words, we consider the space of pairs of functions,

$$(\mathrm{Fun}(M, \mathcal{X}^+), \mathrm{Fun}(M, \mathcal{X}^-)),$$

equipped with the "pointwise product" $(\Pi_r(f,g,h))(x) := \Pi_r(f(x), g(x), h(x))$. Specializing to the case $(\mathcal{X}^+, \mathcal{X}^-) = (\mathbb{KP}^1, \mathbb{KP}^1)$, we get the geometric analog of the usual function spaces $\mathrm{Fun}(M, \mathbb{K})$. The philosophy of non-commutative geometry associates the usual function spaces (which are commutative geometries) to Classical Mechanics, whereas more general non-commutative, but still associative, geometries are associated with Quantum Mechanics. In some sense, the preceding constructions provide a non-associative counterpart to this philosophy—see [10] for a further discussion of this viewpoint.

4.1.4 Faithfulness (Non-degeneracy)

The geometry $(\mathcal{X}^+, \mathcal{X}^-)$ is called *faithful* if $\mathcal{X}^-$ is faithfully represented by its effect of linearizing $\mathcal{X}^+$, and vice versa: whenever $\alpha^\top = \beta^\top$ as sets *and* as linear spaces (with respect to some origin o), then $\alpha = \beta$, and dually. (Faithfulness corresponds to *non-degeneracy* in the Jordan-theoretic sense.) Note that, in a faithful geometry, the component g^+ of an automorphism (g^+, g^-) determines uniquely the second component g^- which must correspond to push-forward of linear structures via g^+.

Example The classical geometries (Grassmann or Lagrangian geometries) are all faithful: the projective group is faithfully represented by its action on $\mathcal{X}^+$. A very degenerate geometry is the *trivial geometry*: take a pair (E, F) of $\mathbb{K}$-modules with trivial transversality relation (all (x,α) are transversal), and all affine structures are the same, equal to the given ones on E, resp. F.

4.1.5 Connectedness and Stability

We will say that two points $x, y \in \mathcal{X}^+$ are *on a common chart* if there is $\alpha \in \mathcal{X}^-$, such that $x, y \in \alpha^\top$. Equivalently, $x^\top \cap y^\top \neq \emptyset$. We will say that $x, y \in \mathcal{X}^+$ are *connected* if there is a sequence of points $x_0 = x, x_1, \ldots, x_k = y$ such that x_i and x_{i+1} are on a common chart. This defines an equivalence relation on $\mathcal{X}^+$ whose equivalence classes are called *connected components of* $\mathcal{X}^+$. By duality, connected components of $\mathcal{X}^-$ are also defined. The geometry is called *connected* if both $\mathcal{X}^+$ and $\mathcal{X}^-$ are connected.

The pair geometry $(\mathcal{X}^+, \mathcal{X}^-, \top)$ will be called *stable* if any two points $x, y \in \mathcal{X}^+$ are on a common chart, and dually for any pair of points $\alpha, \beta \in \mathcal{X}^-$. Clearly, a stable geometry is connected (the converse is not true).

Example In finite dimension over a field, Grassmann and symplectic Lagrange geometries are stable because the affine parts $\alpha^\top$ then are Zariski-dense in $\mathcal{X}^+$, hence have non-empty intersection. By contrast, the *total Grassmann geometry* ($\mathcal{X}^+ = \mathcal{X}^- =$ the set off *all* linear subspaces) is in general highly non-connected (in finite dimension, its connected components are the Grassmannians of subspaces of a fixed dimension).

4.2 Laws

We now describe some more specific laws which may or may not hold in a linear pair geometry. We assume from now on that our geometry is an *affine* pair geometry; the suitable formulation of laws for geometries that are not affine is a completely open problem for the time being. There are two "fundamental laws of projective geometry", denoted in the sequel by (PG1) and (PG2), which can be put in a very concise form as follows and on which we will comment in the sequel:

$$(L^{(r)}_{o,\alpha})^t = L^{(r)}_{\alpha,o}, \tag{PG1}$$

$$(M^{(r)}_{x,y})^t = M^{(r)}_{y,x}. \tag{PG2}$$

The notation L, R, M refers to operators of "left", "right" and "middle translations": the ternary map Π_r gives rise to the operators

$$\Pi_r(x, \alpha, y) =: L^{(r)}_{x,\alpha}(y) =: R^{(r)}_{\alpha,y}(x) =: M^{(r)}_{x,y}(\alpha).$$

Here, $L^{(r)}_{x,\alpha}$ is just the *dilation* denoted before by $r_{x,\alpha}$, and in an *affine* pair geometry we have the simple relation $R^{(1-r)}_{\alpha,y} = L^{(r)}_{y,\alpha}$. Whereas these two operators act on (parts of) $\mathcal{X}^+$, the middle translation $M^{(r)}_{x,y}$ acts from (a part of) $\mathcal{X}^-$ to its "dual" $\mathcal{X}^+$!

4.2.1 The First Law

We say that an affine pair geometry *satisfies the First Law* if (PG1) holds: for all transversal pairs (o, α) and scalars $r \in \mathbb{K}$, the pair of dilations (i.e., left translations) $(r_{o,\alpha}, r_{\alpha,o}) = (L^{(r)}_{o,\alpha}, L^{(r)}_{o,\alpha})$ forms a structural pair of morphisms. In view of the definition of a structural pair in Sect. 4.1.2, this can be written as an identity in 5 variables: for all $r, s \in \mathbb{K}$,

$$\Pi^+_r(o, \alpha, \Pi^+_s(x, \Pi^-_r(\alpha, o, \beta), y)) = \Pi^+_s(\Pi^-_r(o, \alpha, x), \beta, \Pi^+_r(o, \alpha, y)). \tag{PG1}$$

If r is invertible, then the dilation $r_{o,\alpha}$ is invertible on $\alpha^\top$. We require that the pair $(r_{o,\alpha}, r_{\alpha,o})$ extends to a bijection of $(\mathcal{X}^+, \mathcal{X}^-)$; then the identity (PG1) can be interpreted by saying that the pair

$$(g, g') := (r_{o,\alpha}, r^{-1}_{\alpha,o})$$

is an automorphism (called an *inner automorphism*) of the affine pair geometry $(\mathcal{X}^+, \mathcal{X}^-)$. In particular, this means that to every transversal pair (o, α), there is attached a homomorphic image of $\mathbb{K}^\times$, acting by automorphisms of $(\mathcal{X}^+, \mathcal{X}^-)$ and preserving the pair (o, α).

The First Law thus ensures a rich supply of automorphisms, and it is not hard to show that *connected* (PG1)-geometries are homogeneous: the automorphism group $G = \mathrm{Aut}(\mathcal{X}^+, \mathcal{X}^-)$ acts transitively on $\mathcal{X}^+$, on $\mathcal{X}^-$ and on $(\mathcal{X}^+ \times \mathcal{X}^-)^\top$, so that, with respect to a fixed base point (o^+, o^-), we can write

$$\mathcal{X}^+ = G/P^-, \quad \mathcal{X}^- = G/P^+, \quad (\mathcal{X}^+ \times \mathcal{X}^-)^\top = G/H \quad \text{with} \quad H = P^+ \cap P^-.$$

In the non-degenerate finite-dimensional cases over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, G turns out to be a Lie group, and the groups $P^\pm$ are maximal parabolic subgroups of G.

Example The proof that the first law holds, for invertible scalars r, is easy in the case of Grassmann and Lagrangian geometries: it suffices to remark first that the natural action of the projective group $\mathbb{P}\mathrm{Gl}(W)$ on $(\mathrm{Gras}_E^F(W), \mathrm{Gras}_F^E(W))$ is by automorphisms, and second that the automorphism $r\mathrm{id}_E \oplus r^{-1}\mathrm{id}_F$ acts on the space of complements of E by r^{-1} and on those of F by r; put together, this is precisely (PG1). For non-invertible r, the proof is essentially the same (cf. [8]). Lagrangian geometries inherit this law since they are subgeometries of Grassmann geometries.

The First Law should be seen as the "geometric version" of the identity (LJP2) of a Jordan pair: both identities have the same formal structure, with automorphisms and products replaced by derivations and sums, and the inversion of the scalar by the minus-sign in the middle term in (LJP2).

4.2.2 The Second Law

We say that an affine pair geometry *satisfies the Second Law* if (PG2) holds, i.e., if, for all pairs $(x, y) \in \mathcal{X}^+ \times \mathcal{X}^+$ lying on some common chart α (i.e., $\alpha \in x^\top \cap y^\top$), and for all $r, s \in \mathbb{K}$, the pair of "middle translations" $(M_{x,y}^{(r)}, M_{y,x}^{(r)})$ acts as a structural pair between the geometry $(\mathcal{X}^+, \mathcal{X}^-)$ and its dual geometry $(\mathcal{X}^-, \mathcal{X}^+)$. As for the First Law, this is an identity in 5 variables:

$$\Pi_r^+(x, \Pi_s^-(\alpha, \Pi_r^+(y, \beta, x), \gamma), y) = \Pi_s^+(\Pi_r^-(x, \alpha, y), \beta, \Pi_r^+(x, \gamma, y)). \tag{PG2}$$

Of course, we require also the "dual version" of this identity to hold. The scalar $r = \frac{1}{2}$ plays a special rôle with respect to this law, because it satisfies $M_{x,y}^{(r)} = M_{y,x}^{(r)}$, and hence the operator $f = M_{x,y}^{(r)}$ is "self-adjoint" in the sense that $f^t = f$.

Example The proof that the second law holds in a Grassmann geometry is elementary, but considerably more tricky than the proof of the First Law—one may use the explicit formula from Sect. 2.2.2 for the multiplication maps and then prove (PG2) in a suitable affinization, see [8]. In the course of that proof, one sees that (PG2) is indeed the geometric analog of the Fundamental Formula.

4.3 Generalized Projective Geometries

A *generalized projective geometry over* $\mathbb{K}$ is an affine pair geometry over $\mathbb{K}$ such that the First and the Second Law are satisfied *in all scalar extensions of* $\mathbb{K}$. The latter, slightly technical, condition is natural from an algebraist's point of view; it ensures that to every geometry $(\mathcal{X}^+, \mathcal{X}^-)$ one can associate, by scalar extension from $\mathbb{K}$ to the ring $\mathbb{K}[\varepsilon]$ of dual numbers over $\mathbb{K}$, in a functorial way, a new geometry $(T\mathcal{X}^+, T\mathcal{X}^-)$, defined over $\mathbb{K}[\varepsilon]$, and called the *tangent geometry* (cf. [6]). In other words, one may apply some algebraic version of differential calculus. We advise the reader to think for the moment of $(\mathcal{X}^+, \mathcal{X}^-)$ as some kind of smooth manifold, so that the tangent bundle in the usual sense exists; by the usual chain rule of differential calculus, the tangent maps $T\Pi$ then satisfy the same identifies as the structure maps themselves, so that the tangent geometry satisfies again (PG1) and (PG2) over the tangent ring $T\mathbb{K}$ which is nothing but $\mathbb{K}[\varepsilon]$ (see [9] for a justification of this point of view).

Example Grassmann geometries: the tangent geometry of $(\mathrm{Gras}_E^F(W), \mathrm{Gras}_E^F(W))$ is simply $(\mathrm{Gras}_{TE}^{TF}(TW), \mathrm{Gras}_{TE}^{TF}(TW))$, where $TW = W \oplus \varepsilon W$, etc., with $\varepsilon^2 = 0$ is constructed in the same way as the complexification of a real vector space, replacing the condition $i^2 = -1$ by $\varepsilon^2 = 0$. The action of $\mathbb{P}\mathrm{Gl}_{\mathbb{K}}(W)$ is then replaced by the action of $\mathbb{P}\mathrm{Gl}_{\mathbb{K}[\varepsilon]}(TW)$. Even if there is no differentiable structure around, everything behaves like a tangent object should do. Thus Grassmann geometries are indeed generalized projective geometries, and so are Lagrangian geometries.

4.4 Generalized Polar Geometries

A *polarity* is an isomorphism of order 2 of $(\mathcal{X}^+, \mathcal{X}^-)$ onto its dual geometry $(\mathcal{X}^-, \mathcal{X}^+)$ and admitting at least one *non-isotropic point* (see Sect. 2.3). A *generalized polar geometry over* $\mathbb{K}$ is a generalized projective geometry over $\mathbb{K}$ together with a fixed polarity $p := p^+ : \mathcal{X}^+ \to \mathcal{X}^-$, $p^- = p^{-1}$. Thus by definition the set $\mathcal{M} := \mathcal{M}^{(p)}$ of non-isotropic elements x in $\mathcal{X}^+$ is non-empty. The scalar $r = -1$ has the remarkable property that it is its own multiplicative inverse. This property is the key for realizing on the set $\mathcal{M}^{(p)}$ the structure of a *symmetric space*: one deduces from the First Law that then the binary map

$$\mu : \mathcal{M} \times \mathcal{M} \to \mathcal{M}, \quad (x, y) \mapsto \mu(x, y) := \Pi_{-1}(x, p(x), y) = (-1)_{x,p(x)}(y)$$

is well-defined, and that is satisfies the following identities: $\forall x, y, z \in \mathcal{M}$,

$$\mu(x, x) = x, \tag{S1}$$

$$\mu(x, \mu(x, y)) = y, \tag{S2}$$

$$\mu(x, \mu(y, z)) = \mu(\mu(x, y), \mu(x, z)). \tag{S3}$$

In other words, $\mathcal{M}$ is equipped with a family of *symmetries* $\sigma_x = \mu(x, \cdot)$ such that the σ_x are automorphisms of order 2 fixing x. Following Loos [31] we say that $\mathcal{M}$ is a *symmetric space* if, moreover, the map μ is smooth and the tangent map $T_x(\sigma_x)$ is the negative of the identity of the tangent space $T_x\mathcal{M}$. In fact, in our purely algebraic setting, this last property makes sense algebraically because σ_x is nothing but the dilation $(-1)_{x,p(x)}$ which is just the negative of the identity map on the linear space $((p(x))^\top, x)$. By some abuse of language, we thus may call $\mathcal{M} = \mathcal{M}^{(p)}$ the *symmetric space of a generalized polar geometry*. (Under suitable assumptions, this is indeed a smooth symmetric space in the differential geometric sense, see Theorem 4.6.3 below. If, moreover, the dimension is finite over $\mathbb{K} = \mathbb{R}$, then the topological connected components are homogeneous symmetric spaces G/H, see [31].) Since the symmetric space clearly depends functorially on the generalized polar geometry, this correspondence is called the *geometric Jordan-Lie functor* (see [5, 6]). We have already given in Sect. 2.3.3 some examples of symmetric spaces arising in this way.

4.5 Null Geometries

An *absolute null geometry over* $\mathbb{K}$ is a generalized projective geometry over $\mathbb{K}$ together with a fixed absolute null-system $n : \mathcal{X}^+ \to \mathcal{X}^-$ (that means that n commutes with all inner automorphisms of $(\mathcal{X}^+, \mathcal{X}^-)$). Such geometries have several remarkable properties, some of which can be used to give equivalent characterizations (cf. [7]); for instance, they admit "inner polarities":

Example Fix two points $x, y \in \mathcal{X}^+$ and consider the *midpoint map* associating to every affinization $\alpha \in \mathcal{X}^+$ the geometric midpoint $\Pi_{\frac{1}{2}}(x, \alpha, y) \in \mathcal{X}^+$ of x and y in the affine part defined by α. Null geometries are characterized by the remarkable property that x and y can be chosen such that every "generic" point can appear as geometric midpoint of x and y. Then the midpoint map extends to a bijection from $\mathcal{X}^-$ onto $\mathcal{X}^+$, and the Second Law now implies that this bijection is a polarity, called an *inner polarity*. Just to see how surprising this property is, consider the case of ordinary projective geometry: here the geometric midpoint of x and y always lies on the projective line spanned by x and y, hence has a rather special and non-generic position. However, if the projective space itself was already a projective line, then indeed every point different from x and y can play the role of a midpoint of x and y. This corresponds to the fact, already mentioned in the examples of Sect. 2.3.2, that among the usual projective spaces only the projective line is an absolute null geometry.

4.6 The Jordan Functor

Recall that the *Lie functor* describes the correspondence between Lie groups and Lie algebras. The Jordan analog of this correspondence is described by the following result; in contrast to the Lie functor, the correspondence works equally well in arbitrary dimension and over general base fields and rings (we only have to assume that 2 and 3 are invertible in $\mathbb{K}$).

Theorem 4.6.3 (The Jordan functor) *There are correspondences* (*essentially bijections*) *between the following objects*:

(1) *connected generalized projective geometries* $(\mathcal{X}^+, \mathcal{X}^-)$ *with base point* (o^+, o^-), *and Jordan pairs* (V^+, V^-);
(2) *connected generalized polar geometries* $(\mathcal{X}^+, \mathcal{X}^-, p)$ *with base point* o, *and Jordan triple systems* V;
(3) *connected absolute null geometries* $(\mathcal{X}^+, \mathcal{X}^-, n)$ *with base point* o, *and Jordan algebras* V *admitting a unit element* e.

Moreover, these correspondences are functorial in both directions. (*To be more precise, we have two functors* D *"differentiating at the base point" and* I *"integrating" such that* $D \circ I$ *is the identity*; *under certain restrictions such as finite dimensionality over a field*, D *and* I *are indeed equivalences of categories*.) *Moreover, under these correspondences*,

(i) *the* (*algebraic*) *Jordan-Lie functor from Sect*. 3.6 *and the geometric Jordan-Lie functor correspond to each other*;
(ii) *inner ideals correspond to intrinsic subspaces containing the base point*.

Finally, assume $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *or any other topological ring having a dense unit group. Then* $\mathcal{X}^+$ *and* $\mathcal{X}^-$ *are smooth manifolds over* $\mathbb{K}$ *if* (*and only if*) *the Jordan pair satisfies a certain regularity condition* (*called a* continuous quasi inverse Jordan pair *in* [14]). *If we add a* *continuous polarity p to these data, then the corresponding symmetric space $\mathcal{M}^{(p)}$ is an open submanifold of $\mathcal{X}^+$, and its symmetric space structure is smooth.*

Example Under the correspondence from the theorem,

- Grassmann geometries $(\mathrm{Gras}_E^F(W), \mathrm{Gras}_F^E(W))$ correspond to the Jordan pair of "rectangular matrices" $(\mathrm{Hom}(E, F), \mathrm{Hom}(F, E))$. If $E \cong F$, then this is a null geometry corresponding to the Jordan algebra $\mathrm{End}(E)$.
- Lagrangian geometries of a symplectic form are null geometries and correspond to Jordan algebras of symmetric matrices.
- Lagrangian geometries of a symmetric neutral form correspond to Jordan pairs of skew-symmetric matrices. They are null-geometries if these matrices are of even size $(2n \times 2n)$. For $\mathbb{K} = \mathbb{C}$, Lagrangian geometries of a Hermitian neutral form correspond to Jordan pairs of skew-Hermitian matrices; they are always null geometries, corresponding to the isomorphism between skew-Hermitian and Hermitian matrices and to the Jordan algebra structure on the latter.
- Projective quadrics carry a structure of a null geometry that corresponds to the spin-factor (see Item (4) in the Example of 3.1).

Remarks about the Theorem and its Proof As for the correspondence between Lie groups and Lie algebras, one, has to give two constructions that are inverse to each other: starting with a geometry, we get the associated infinitesimal object by "differentiation". For Lie groups, differentiating once, one simply gets the tangent space without any useful information (the Lie algebra $\mathfrak{g}$ with addition, but no bracket); differentiating ones, one gets the Lie bracket, and finally one needs a third order argument to prove the Jacobi identity. In the present case the situation is quite similar (see [6]).

The inverse construction is fairly easy: roughly, starting from a Jordan pair (V^+, V^-), one uses the Kantor-Koecher-Tits construction described in Sect. 3.2 to define the associated 3-graded Lie algebra $\mathfrak{g}$. Since elements of $\mathrm{ad}(\mathfrak{g}_1)$ and $\mathrm{ad}(\mathfrak{g}_{-1})$ are 2-step nilpotent, their exponential is just a quadratic polynomial, so that we have well-defined groups $U^\pm = \exp(\mathrm{ad}(\mathfrak{g}_{\pm 1}))$. The subgroup $G = \langle U^+, U^- \rangle$ of $\mathrm{Aut}(\mathfrak{g})$ generated by these two groups is called the *elementary projective group of the Jordan pair*. Let $H \subset G$ be the stabilizer of the 3-grading ("diagonal matrices in G") and $P^\pm := HU^\pm$ ("parabolics"; this is a semidirect product). Then $(\mathcal{X}^+, \mathcal{X}^-) = (G/P^-, G/P^+)$, with transversality defined by the G-orbit of the canonical base point (eP^-, eP^+), is the geometry sought for. The hard part is now to verify that (PG1) and (PG2) indeed hold and that the construction is functorial (see [6]).

Part (3) on null geometries is proved in [7], and statement (ii) on intrinsic subspaces in [12]. The final statement on smooth structures has been obtained in joint work with K.-H. Neeb [13, 14]; see also [10]. The setting of differential calculus, smooth manifolds and Lie groups over topological base fields and rings has been developed in [15]—see also [9], Sect. 2, for the basic facts; we hope to convince the reader that the resulting theory really is much simpler than "usual" differential calculus in Banach or Fréchet spaces. It covers, in particular, the interesting cases $\mathbb{K} = \mathbb{Q}_p$ (the p-adic numbers), $\mathbb{K} = \mathbb{R} \times \mathbb{R}$ (para-complex numbers) and $\mathbb{K} = \mathbb{R}[\varepsilon]$ (dual numbers). An approach to differential geometry, Lie groups and symmetric spaces in this general framework is worked out in [9]; much of the algebraic framework developed there has been designed aiming at applications in situations like the ones discussed here.

Example A *Banach Jordan pair* is a pair of Banach spaces with a continuous trilinear Jordan pair structure. In this case the conditions from [14] are always fulfilled, so that the corresponding geometry $(\mathcal{X}^+, \mathcal{X}^-)$ is indeed a pair of smooth (Banach-)manifolds. (This can also be proved by more conventional functional analytic methods, see the monograph [40].) For instance, full or Hermitian algebras of continuous operators are Banach Jordan pairs, and hence the geometry associated to $\mathrm{Herm}(\mathcal{H})$ is a smooth manifold, in fact isomorphic to $\mathrm{U}(\mathcal{H})$. There is a huge literature on Banach Jordan structures, see references in [1, 24, 41].

5 Comments and Prospects

As said in the introduction, the theory exposed in this paper is purely mathematical; the description by using terms borrowed from the language of physics may be seen as a game without any relation to the "real world". Be this as it may, I would like to put forward some arguments why I think that it is interesting to play this game and maybe to pursue it even further:

(1) First of all, it is certainly not the only, but at least one possible interpretation of the "Jordan Ansatz", and hence it does cover the standard setting of quantum mechanics (the Jordan algebra $\mathrm{Herm}(\mathcal{H})$): thus it should have *some* meaning, be it relevant or not.
(2) Apart from the original motivation by the Jordan Ansatz, our setting incorporates other viewpoints that have shown up in the search for foundations of quantum mechanics:

- the fundamental rôle of projective geometries ([42], p. 6: "… quantum mechanical systems are those whose logics form some sort of projective geometries", [16]: "… any specific feature of projective geometry gives rise to a physically realizable characteristic of quantum mechanics");
- linearity ([2]: "Perhaps the habitual linear structures of quantum mechanics are analogous to the inertial rest frames in special relativity …");
- duality ([36], p. 527: "The pure philosopher may start from a postulated unity and call it Being. He may then concede the necessity of distinguishing two modes of being and call them reality and logos or whatever else … A physicist groping with his science is after all following the same path …");
- quantum non-locality (see the comparison with twistor theory in Sect. 2.4.2);
- Hermitian symmetric spaces (see, e.g., the paper "The pure state space of quantum mechanics as Hermitian symmetric space" [19]. One should not underestimate the fact that modern Jordan theory is intimately related to the theory of finite and infinite dimensional Hermitian symmetric spaces, by work of Koecher, Loos, Kaup, Upmeier and others; cf. [28, 29, 40] and the "Colloquial Survey of Jordan Theory" in [34]. This aspect has not been noticed at all in [19]).
- the search for using "exceptional" groups and geometries in physics (Jordan theory is deeply mixed into the structure of Freudenthal's "magic square" which is a main source of exceptional geometries; cf. also references in [24]).

If one strives for unity of mathematics, it is very satisfying to realize that all these aspects can be incorporated, without mutilating any of them, in a common framework.

(3) I consider "non-associative geometry" as some sort of natural counterpart of "noncommutative geometry" (see Sect. 4.1.3 and [10]; as usual "non" means here: "not

necessarily").[1] As I tried to explain in my discussion of the quotation from [1], even if at the end we will be forced to return to some "associative geometry" for describing quantum mechanics, the decomposition of the associative product in its Jordan and Lie part somehow seems to correspond to the fundamental problem of the coexistence of the **U**-evolution and the state reduction **R**. Therefore it might be useful to widen the scope from associative to non-associative structures.

(4) Be our universe unique or not, a mathematician has the natural desire to look at it as belonging to a *category*, so that usual categorial notions should apply to it. This means that he would like to understand the universe by its *properties* and not by a construction such as "take an infinite dimensional separable Hilbert space and do this and that ...". We encounter the same problem already on the level of ordinary projective spaces: one can define them by the usual construction ("take the rays in a vector space ..."), or by intrinsic properties—when doing the latter, all modern authors more or less follow the famous model of Hilbert's "Grundlagen der Geometrie" where the *incidence axioms* of projective geometry were shown to be the intrinsic geometric properties sought for. We simply propose to replace here "incidence axioms" by other foundational properties, namely by "laws", in the sense of Sects. 2.2.2 and 4.2. It turns out that, unlike incidence structures, algebraic laws are very flexible and can be adapted to a great variety of situations. For instance, notions of *direct products* and *bundles* (see below) exist in our setting, whereas they are in general not compatible with interesting incidence axioms (the direct product of two ordinary projective spaces is no longer an ordinary projective space!). In some sense, it seems to me that this is the deeper mathematical reason why the incidence geometric approach to quantum mechanics, due to Birkhoff and von Neumann, has been gradually abandoned, in spite of its great mathematical beauty (see the book [42]).

(5) I already mentioned (Sect. 2.5) the problem that "tensor products of Jordan algebras" do not exist and hence it is not clear at all how many-particle systems should be modelized in our geometric approach. However, thanks to the flexibility of algebraic laws just mentioned, the situation is not as hopeless as it might look. For several reasons, it seems to me that the suitable geometric setting replacing the tensor product construction from usual quantum mechanics should be related to a notion of *vector bundles in the category of generalized projective geometries*. In fact, "vector bundles in the category of symmetric spaces" have been introduced and studied in [20] and [11]: essentially, a vector bundle in some geometric category is a vector bundle F over a base M such that both F and M are objects of the category and such that some natural compatibility conditions are satisfied. On the infinitesimal level, the corresponding notion is the one of *general representation* or $\mathfrak{m}$-*module* for the tangent object $\mathfrak{m}$ of the base M. For instance, in the category of Lie groups we get the usual notion of representation or G-module of the base G. Now, similarly as in the category of Lie groups, for symmetric spaces and generalized projective geometries, there exist notions replacing tensor products for such representations. As a variant of this, one may also leave the category of vector bundles and modelize many-particle systems by *multilinear bundles* which have been introduced in [9] precisely with the aim to replace tensor products of vector bundles by more geometric notions. In any case, it is tempting, by employing the language of $\mathfrak{m}$-representations, to associate "atoms" with simple geometries or irreducible

[1] As I learned later, the term "non-associative geometry" has previously been introduced by L.V. Sabinin [35] in the more specific context of *quasigroups* and *loops*. Symmetric spaces are prominent examples of such structures, but as far as I know, Jordan algebraic structures cannot be interpreted in this context.

representations, "molecules" with suitable extensions of simple geometries, and, on the opposite end, to interpret "classical mechanics" via function geometries, that is, certain continuous direct products of the standard fiber which has not much interesting internal structure.

(6) The fact that we may think of generalized projective geometries both classically and quantum-mechanically is rather puzzling (see example in Sect. 2.2.3): the conformal compactification of Minkowski-space as well as the geometry of quantum mechanics are examples of generalized projective geometries. Is this an accident? Some speculations about this question can be found in [18].[2]

(7) The mathematical similarity between compactified Minkowski space and the generalized projective geometry of quantum mechanics suggests that it may be necessary to carry these ideas even further and to go beyond the category of generalized projective geometries: namely, comparing with the historical development of theories from Newtonian mechanics via Special Relativity to General Relativity, Newtonian mechanics would correspond to the standard Hilbert space formulation of quantum theory, Special Relativity (in compactified Minkowski space) would correspond to our hypothetical "generalized projective" formulation, and hence General Relativity should correspond to an even more hypothetical generalization in terms of geometries that are "modelled on generalized projective geometries", but are no longer "conformally flat" (in a suitable sense; recall that compactified Minkowski space is conformally flat). In finite dimension, geometries modelled on certain homogeneous spaces G/P have been studied by Elie Cartan using what is nowadays called a *Cartan connection* (which generalizes the projective and conformal connections; see [39] for a modern presentation). As far as I know, Jordan theory or more general non-associative algebra have not yet been systematically used as an approach to study the corresponding Cartan geometries, but certainly this would open the way for a theory of their infinite-dimensional generalization. Summing up, if the analogies mentioned above have some meaning, it could be hoped that Jordan geometry gives some hints on what the last two items in the following matrix might be:

geometry:	linear; affine	projective	manifold
mechanics:	classical	special relativistic	general relativistic
quantum theory:	Hilbert space q.m.	projective q.m. ?	Cartan geometric q.m. ??

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[2] For a popular account see p. 404 in: von Weizsäcker, C.F.: Aufbau der Physik. Hanser, München (1985).

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